# §6.3 Orthogonal Projections

Last time we saw how to project a vector onto a line. What if we want a projection onto a plane? Or in general some span of vectors in $\mathbb{R}^n$?

## Definition

Let $W$ be a subspace of $\mathbb{R}^n$. The orthogonal complement of $W$ is the set of vectors orthogonal to every vector in $W$. We notate it as $W^\perp$. In others words

$$u \cdot v = 0$$

for any $u$ in $W$ and $v$ in $W^\perp$

## Example

Suppose $\{v_1, \dots, v_n\}$ is an orthogonal basis of $\mathbb{R}^n$ and let $W = \text{span}\{v_1, \dots, v_k\}$ for $k \leq n$.

Then $W^\perp = \text{span}\{v_{k+1}, \dots, v_n\}$

## Theorem (Orthogonal Decomposition Theorem)

Let $W$ be a subspace of $\mathbb{R}^n$. Then any vector $y$ in $\mathbb{R}^n$ can be written uniquely as

$$y = \hat{y} + z$$

where $\hat{y}$ is in $W$ and $z$ is in $W^{\perp}$. Here $\hat{y}$ is called the <u>projection of $y$ onto $W$</u> and is sometimes written as $\hat{y} = \text{proj}_W y$

How do we find $\hat{y}$ and $z$?

If $\{u_1, \dots, u_m\}$ is an orthogonal basis of $W$, then

- $\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2 + \cdots + \left(\frac{y \cdot u_m}{u_m \cdot u_m}\right) u_m$

- $z = y - \hat{y}$

Notice this generalizes the case from last time when projecting onto $L = \text{span}\{u\}$.

## Proof

Since $\hat{y}$ is in $W$ and $\{u_1, u_2, ---, u_m\}$ is an orthogonal basis for $W$, the theorem from §6.2 tells us

$$\hat{y} = \left(\frac{\hat{y}\cdot u_1}{u_1\cdot u_1}\right)u_1 + \left(\frac{\hat{y}\cdot u_2}{u_2\cdot u_2}\right)u_2 + ---- + \left(\frac{\hat{y}\cdot u_m}{u_m\cdot u_m}\right)u_m$$

which is almost what we want. It suffices to show $\hat{y}\cdot u_i = y\cdot u_i$ for $i=1,---,m$ to obtain the result.

Since $z = y-\hat{y}$ is orthogonal to $W$, we have

$$0 = z\cdot u_i = (y-\hat{y})\cdot u_i = y\cdot u_i - \hat{y}\cdot u_i$$

so indeed $y\cdot u_i = \hat{y}\cdot u_i$ for all $i=1,---,m$

Thus

$$\hat{y} = \left(\frac{y\cdot u_1}{u_1\cdot u_1}\right)u_1 + \left(\frac{y\cdot u_2}{u_2\cdot u_2}\right)u_2 + ---- + \left(\frac{y\cdot u_m}{u_m\cdot u_m}\right)u_m$$

## Example

Let $W$ be the subspace of $\mathbb{R}^3$ spanned by

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

a) Verify $\{u_1, u_2\}$ is an orthogonal basis of $W$

b) Write $y = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$ as a sum of a vector in $W$ and a vector in $W^\perp$.

## Solution

a) This is clear.

b) $y = \hat{y} + z$ where

$$\hat{y} = \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right) u_2$$

$$= \left(\frac{-2+6+3}{1+1+1}\right) u_1 + \left(\frac{-6-12-3}{9+4+1}\right) u_2$$

$$= \frac{7}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \underbrace{\left(\frac{-21}{14}\right)}_{= -3/2} \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -13/6 \\ 16/3 \\ 23/6 \end{bmatrix}$$ after getting common denominators

Now $z = y - \hat{y} = \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} -13/6 \\ 16/3 \\ 23/6 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 2/3 \\ -5/6 \end{bmatrix}$

Theorem (Best Approximation Theorem)

With the notation as before if $W$ is a subspace of $\mathbb{R}^n$, $y$ any vector in $\mathbb{R}^n$, and $\hat{y} = \text{proj}_W y$ then $\hat{y}$ is the ~~clos~~ vector in $W$ closest to $y$. In other words

$$\| y - \hat{y} \| < \| y - v \|$$

for all $v$ in $W$ with $v \neq \hat{y}$

Thus, just as in §6.2, $\|z\| = \|y - \hat{y}\|$ is the shortest distance from $y$ to $W$.

Remark

Notice $y$ is in $W$ if and only if $y = \hat{y} = \text{proj}_W y$. This of course implies $z = 0$ so the distance of $y$ to $W$ is zero.

The previous result requires only an orthogonal basis for $W$. The result is even nicer when an orthonormal basis is selected.

Theorem

Let $\{u_1, \dots, u_m\}$ be an orthonormal basis of subspace $W$ of $\mathbb{R}^n$. For any vector $y$ in $\mathbb{R}^n$,

$$\text{proj}_W y = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \cdots + (y \cdot u_m)u_m$$

In particular, if $U = [u_1 | u_2 | \cdots | u_m]$ is the $n \times m$ matrix with columns $u_1, \dots, u_m$, then also

$$\text{proj}_W y = UU^T y$$

Proof

The first statement follows from the previous formula as $u_i \cdot u_i = 1$ for $i = 1, \dots, m$.

The second statement follows as the entries of $U^T y$ are $u_1^T y, u_2^T y, \dots, u_m^T y$ which are exactly $u_1 \cdot y, u_2 \cdot y, \dots, u_m \cdot y$

## Example

Suppose $\{u_1, u_2\}$ is an orthonormal basis for $W$ $= \operatorname{span}\{u_1, u_2\}$

where

$$u_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

a) check this is an orthonormal basis

b) compute $\text{proj}_W y$ where $y = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$

## solution

a) $\|u_1\| = \|u_2\| = 1$ and $u_1 \cdot u_2 = 0$

b)
$$\text{proj}_W y = \begin{bmatrix} 1/3 & -2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/9 & 4/9 & -2/9 \\ 4/9 & 5/9 & 2/9 \\ -2/9 & 2/9 & 8/9 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 41/9 \\ 49/9 \\ 16/9 \end{bmatrix}$$